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Integrable potentials with logarithmic integrals of motion

S Ichtiaroglou and G Voyatzis

Department of Physics, University of Thessaloniki, Thessaloniki, Greece

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Abstract. In this paper several examples of integrable potentials with a second invariant which contains a logarithmic part in the velocities are given. In most cases the potential depends on the velocities. There is, however, a case where the potential is velocity independent.

1. Introduction

Direct construction of planar integrable potentials possessing an integral of motion of a prescribed form can be obtained by a method due to Bertrand (1852). Darboux (1901) obtained the general form for a velocity-independent potential in order to possess a second invariant, quadratic in the velocities. These results are also presented in Whittaker (1937, p 332). Dorizzi *et al* (1983) and Ankiewicz and Pask (1983) completed the above results for a case not considered by the former authors.

Polynomial integrals of motion of degree greater than 2 for planar systems have been found by this method and by other methods as well and the results in this field can be found in a review by Hietarinta (1986).

Bozis and Ichtiaroglou (1987) attacked the problem from a different point of view, i.e. they found necessary and sufficient conditions in order for a given function of position and velocity to be an integral of motion of a planar force field not given in advance. The case of velocity-dependent potentials has also been studied and in the generic case, if the given function satisfies certain conditions, the corresponding force field is determined uniquely.

Integrable velocity-dependent potentials which possess linear or quadratic invariants in the velocity have been found by Dorizzi *et al* (1985). Very few, however, are the results concerning transcendental invariants. A few examples are given in Hietarinta (1984), among which is included a logarithmic invariant. More systematically, transcendental invariants which are arbitrary functions of two different polynomials in velocities are studied in Hietarinta (1986, pp 70–83) where, in two cases, logarithmic invariants are found.

In this paper we construct integrable planar potentials which possess a second integral of motion of the form

$$I = F_1\dot{x} + F_2\dot{y} + F_3 + \ln|G_1\dot{x} + G_2\dot{y} + G_3|$$

where F_i and G_i are functions of x, y. We do not attempt to find all systems possessing an invariant of the above form. However we offer several examples of such systems. Since function I does not have a definite parity under time reflection, most potentials are velocity dependent, i.e. they correspond to motion in a rotating system or under the influence of electromagnetic forces.

2. General remarks on the form of the invariant

We consider a particle of unit mass moving on the xy plane under the influence of the velocity-dependent potential

$$V = U(x, y) + A(x, y)\dot{x} + B(x, y)\dot{y}. \tag{1}$$

Only linear terms in the velocity are included in (1), since otherwise the given forces would be functions of the accelerations, which is inadmissible in Newtonian dynamics (e.g. Pars 1965, p 82).

The corresponding accelerations are given by

$$\ddot{x} = -U_x + \Omega\dot{y} \tag{2a}$$

$$\ddot{y} = -U_y - \Omega\dot{x} \tag{2b}$$

where subscripts denote partial differentiation, and

$$\Omega(x, y) = A_y - B_x. \tag{3}$$

Because of the gauge invariance of (2), functions A and B in (1) cannot be determined uniquely, so we consider that the velocity-dependent potential V is determined if functions $U(x, y)$ and $\Omega(x, y)$ are defined.

We suppose that system (1), besides the energy integral

$$E = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + U(x, y) \tag{4}$$

possesses a second invariant of the form

$$I = F_1\dot{x} + F_2\dot{y} + F_3 + \ln|G_1\dot{x} + G_2\dot{y} + G_3| \tag{5}$$

where F_i, G_i are functions of x, y. In order that I is an integral of motion of system (1), it must hold identically:

$$\mathrm{d}I/\mathrm{d}t = I_x\dot{x} + I_y\dot{y} + I_{\dot{x}}(-U_x + \Omega\dot{y}) + I_{\dot{y}}(-U_y - \Omega\dot{x}) \equiv 0. \tag{6}$$

Equation (6) yields the equations

$$G_1F_{1x} = 0 \tag{7a}$$

$$G_2F_{1x} + G_1(F_{2x} + F_{1y}) = 0 \tag{7b}$$

$$G_1F_{2y} + G_2(F_{2x} + F_{1y}) = 0 \tag{7c}$$

$$G_2F_{2y} = 0 \tag{7d}$$

$$G_{1x} + G_3F_{1x} + G_1(F_{3x} - \Omega F_2) = 0 \tag{8a}$$

$$(G_{2x} + G_{1y}) + G_3(F_{2x} + F_{1y}) + G_2(F_{3x} - \Omega F_2) + G_1(F_{3y} + \Omega F_1) = 0 \tag{8b}$$

$$G_{2y} + G_3F_{2y} + G_2(F_{3y} + \Omega F_1) = 0 \tag{8c}$$

$$-G_1(F_1U_x + F_2U_y) + G_3(F_{3x} - \Omega F_2) + (G_{3x} - \Omega G_2) = 0 \tag{9a}$$

$$-G_2(F_1U_x + F_2U_y) + G_3(F_{3y} + \Omega F_1) + (G_{3y} + \Omega G_1) = 0 \tag{9b}$$

$$G_3(F_1U_x + F_2U_y) + (G_1U_x + G_2U_y) = 0. \tag{10}$$

Since the logarithmic part in (5) is always supposed to contain velocity-dependent terms (otherwise I becomes merely linear in the velocities), at least one of the functions G_1 and G_2 must be different from zero. In this case, equations (7) yield the solution

$$F_1 = \delta y + \varepsilon \tag{11a}$$

$$F_2 = -\delta x + \zeta \tag{11b}$$

where δ, ε, ζ are arbitrary constants.

At this point we observe that, as long as functions G_i are not determined, function F_3 can be chosen arbitrarily, which is equivalent to the extraction of a suitable common factor out of the logarithmic part of I. The polynomial in the logarithmic part can also be divided by any constant, since it corresponds to addition of a constant to I.

We select F_3 such that

$$F_{3x} = \Omega F_2 \tag{12a}$$

$$F_{3y} = -\Omega F_1. \tag{12b}$$

If we take into account equations (11), from (12) we obtain

$$F_3 = F_3(u) \tag{13}$$

and

$$\Omega = \Omega(u) = -F_{3u} \tag{14}$$

where

$$u = \tfrac{1}{2}\delta(x^2 + y^2) + \varepsilon y - \zeta x. \tag{15}$$

It is understood that (12) also restrict the form of Ω, since it must be a function of u.

Equations (8), with the help of (11) and (12), take the form

$$G_{1x} = 0 \tag{16a}$$

$$G_{2x} + G_{1y} = 0 \tag{16b}$$

$$G_{2y} = 0 \tag{16c}$$

with the solution

$$G_1 = \alpha y + \beta \tag{17a}$$

$$G_2 = -\alpha x + \gamma \tag{17b}$$

where α, β, γ are arbitrary constants. Taking into account (12), equations (9) take the form

$$G_{3x} = \Omega G_2 + G_1(F_1 U_x + F_2 U_y) \tag{18a}$$

$$G_{3y} = -\Omega G_1 + G_2(F_1 U_x + F_2 U_y). \tag{18b}$$

Functions $G_3(x, y)$, $F_3(u)$ and $U(x, y)$ are still undetermined, while Ω can be found from (14).

Equations (18) and (10) are too complicated to be treated in their general form. In the next section several examples will be given for special cases.

3. Examples of logarithmic integrals of motion

In this section we find particular solutions of (10) and (18) for some special values of the constants δ, ε, ζ and α, β, γ. In all cases we take $\varepsilon = 1$, but the results can easily be generalised also for $\varepsilon \neq 1$.

(a) Case $\delta = \alpha = \beta = 0$

In this case we can put $\gamma = 1$ and, performing a suitable rotation of the coordinate system, ζ can be put equal to 0. From (15) we have $u = y$ and I takes the form

$$I = \dot{x} + F_3(y) + \ln|\dot{y} + G_3| \tag{19}$$

while (18) and (10) become

$$G_{3x} = \Omega(y) \tag{20a}$$

$$G_{3y} = U_x \tag{20b}$$

and

$$G_3 U_x + U_y = 0. \tag{21}$$

From (20) we get

$$G_3 = x\Omega(y) + f(y) \tag{22}$$

while combining (21) with (20b) we find

$$U_y = -G_3 G_{3y} \tag{23}$$

that is

$$U = -\tfrac{1}{2}G_3^2 + g(x) \tag{24}$$

where f and g are arbitrary functions of y and x, respectively.

Taking into account (22) and (24), (20b) yields

$$\Omega^2 + \Omega_y = k \tag{25a}$$

$$\Omega f + f_y = l \tag{25b}$$

$$g_x = kx + l \tag{25c}$$

where k, l are arbitrary constants. Equation (25c) gives

$$g = \tfrac{1}{2}kx^2 + lx. \tag{26}$$

The additive constant in g is merely an additive constant in V so it is put equal to zero. In the following we consider the two cases $k = 0$ and $k \neq 0$.

(a1) Subcase $k = 0$

In this case (25a), after a suitable translation with respect to the y axis is performed, gives

$$\Omega = 1/y. \tag{27}$$

By (14) we take

$$F_3 = -\ln|y| \tag{28}$$

and after solving (25) and performing a suitable translation with respect to the x axis, equations (22) and (23) give

$$G_3 = x/y + ly/2 \tag{29}$$

and

$$U = -\tfrac{1}{2}(x/y + ly/2)^2 + lx. \tag{30}$$

(a2) Subcase $k \neq 0$

By a suitable translation with respect to the x axis, we can put $l = 0$. Ω and f can be obtained from (25a) and (25b), respectively, and, using (22), (23) and (14), after a suitable translation with respect to the y axis, we take for $k = \lambda^2 > 0$

$$\Omega = \lambda \tanh(\lambda y)$$
$$F_3 = -\ln \cosh(\lambda y)$$
$$G_3 = [\lambda x \sinh(\lambda y) + \mu]/\cosh(\lambda y)$$
$$U = -\tfrac{1}{2}G_3^2 + \tfrac{1}{2}\lambda^2 x^2$$

and for $k = -\lambda^2 < 0$

$$\Omega = -\lambda \tan(\lambda y)$$
$$F_3 = -\ln|\cos(\lambda y)|$$
$$G_3 = [-\lambda x \sin(\lambda y) + \mu]/\cos(\lambda y)$$
$$U = -\tfrac{1}{2}G_3^2 - \tfrac{1}{2}\lambda^2 x^2$$

where λ, μ are arbitrary constants.

(b) Case $\delta = \alpha = 0, \beta \neq 0$

Again, by a suitable rotation of the axes ζ can be put equal to 0, so that $u = y$. In this case I has the form (β can be chosen equal to 1)

$$I = \dot{x} + F_3(y) + \ln|\dot{x} + \gamma\dot{y} + G_3|$$

and (18) and (10) become, respectively,

$$G_{3x} = \gamma\Omega(y) + U_x \tag{31a}$$

$$G_{3y} = -\Omega(y) + \gamma U_x \tag{31b}$$

and

$$(G_3 + 1)U_x + \gamma U_y = 0. \tag{32}$$

From (31a) we obtain

$$U = G_3 - \gamma\Omega(y)x + g(y) \tag{33}$$

while by combining (31a) and (31b) we get the equation

$$\gamma G_{3x} - G_{3y} = \Omega(y)(1 + \gamma^2)$$

with the solution

$$G_3 = (1 + \gamma^2)F_3 + f(\gamma y + x) \tag{34}$$

where f and g are arbitrary functions and (14) has also been taken into account. Functions $\Omega(y)$, $g(y)$ and $f(\gamma y+x)$ must be such that (32) holds. We will not find all possible solutions but we will find examples in three subcases, namely $F_3=0, f=0$ and $f=\gamma y+x$.

(b1) Subcase $F_3 = 0$

In this case we also have $\Omega=0$. Equation (32), taking into account (33) and (34), becomes

$$f'(f+1+\gamma^2) = -\gamma g_y = k \tag{35}$$

where a prime denotes differentiation with respect to $\gamma y+x$ and k is an arbitrary non-zero constant (case $k=0$ is trivial).

From (35), after performing suitable translations, we get

$$g=-ky/\gamma$$

and

$$f=(1+\gamma^2)\left[-1\pm\left(1+\frac{2k(\gamma y+x)}{(1+\gamma^2)^2}\right)^{1/2}\right]$$

which determine completely functions U and G_3.

(b2) Subcase $f = 0$

Equation (32) becomes

$$-\gamma x\Omega_y = \Omega[1+(1+\gamma^2)(1+F_3)]-g_y$$

which gives

$$\Omega = k = \text{constant}$$

i.e.

$$F_3=-ky$$

and

$$g=-\tfrac{1}{2}k^2(1+\gamma^2)y^2+k(2+\gamma^2)y.$$

From (33) and (34) we get

$$U=k(y-\gamma x)-\tfrac{1}{2}k^2(1+\gamma^2)y^2$$

and

$$G_3=-k(1+\gamma^2)y.$$

(b3) Subcase $f = \gamma y + x$

If we differentiate (32) with respect to x, taking into account (33) and (34), we obtain

$$\gamma^2\Omega_y+\gamma\Omega-1=0$$

with the solution

$$\Omega=[1-\exp(-y/\gamma)]/\gamma$$

where again a suitable parallel translation of the y axis has been performed. From (14) we get

$$F_3 = -y/\gamma - \exp(-y/\gamma)$$

while (32) gives

$$g = (1+\gamma^2 - y/\gamma)\exp(-y/\gamma) - \tfrac{1}{2}(1+\gamma^2)\exp(-2y/\gamma) + y/\gamma.$$

Equations (33) and (34) now become

$$G_3 = -y/\gamma + x - (1+\gamma^2)\exp(-y/\gamma)$$

and

$$U = (x - y/\gamma)\exp(-y/\gamma) - \tfrac{1}{2}(1+\gamma^2)\exp(-2y/\gamma).$$

(c) Case $\delta = 0$, $\alpha \neq 0$

In this case we can put $\alpha = 1$ and by a suitable translation and rotation, $\beta = \gamma = \zeta = 0$. The integral I has the form

$$I = \dot{x} + F_3(y) + \ln|\dot{x}y - x\dot{y} + G_3|$$

and (18) become

$$G_{3x} = -x\Omega(y) + yU_x \tag{36a}$$

$$G_{3y} = -y\Omega(y) - xU_x. \tag{36b}$$

Combining (36) we get

$$xG_{3x} + yG_{3y} = -(x^2+y^2)\Omega(y)$$

with the solution

$$G_3 = -(1 + x^2/y^2)\int y\Omega(y)\,\mathrm{d}y + f(x/y)$$

while from (36a) we obtain

$$U = G_3/y + \Omega(y)x^2/2y + g(y)$$

where f, g are arbitrary functions.

In order to find an example for this case, we select

$$\Omega = 1/y$$

i.e.

$$F_3 = -\ln|y|.$$

The remaining equation (10) takes the form

$$(1/w)f\,\mathrm{d}f(w)/\mathrm{d}w = y^3\,\mathrm{d}g(y)/\mathrm{d}y$$

where $w = x/y$, which yields

$$f = (k^2 + c^2x^2/y^2)^{1/2} \qquad g = -c^2/2y^2$$

where k and c are arbitrary constants, so I and U take the form

$$I = \dot{x} + \ln|\dot{x} - x\dot{y}/y + (c^2x^2 + k^2y^2)^{1/2}/y^2 - x^2/y^2 - 1|$$

$$U = -(x^2+c^2)/2y^2 + (c^2x^2 + k^2y^2)^{1/2}/y^2.$$

Table 1. Examples of integrable potentials with logarithmic invariants. γ, λ, μ, k, l and c are arbitrary constants.

	F_3	G_3	I	Ω	U
I	$-\ln\lvert y\rvert$	$x/y+ly/2$	$\dot{x}+F_3+\ln\lvert\dot{y}+G_3\rvert$	$1/y$	$-\frac{1}{2}G_3^2+lx$
II	$-\ln\cosh(\lambda y)$	$\dfrac{\lambda x\sinh(\lambda y)+\mu}{\cosh(\lambda y)}$	$\dot{x}+F_3+\ln\lvert\dot{y}+G_3\rvert$	$\lambda\tanh(\lambda y)$	$-\frac{1}{2}G_3^2+\frac{1}{2}\lambda^2x^2$
III	$-\ln\lvert\cos(\lambda y)\rvert$	$\dfrac{-\lambda x\sin(\lambda y)+\mu}{\cos(\lambda y)}$	$\dot{x}+F_3+\ln\lvert\dot{y}+G_3\rvert$	$-\lambda\tan(\lambda y)$	$-\frac{1}{2}G_3^2-\frac{1}{2}\lambda^2x^2$
IV	0	$(1+\gamma^2)\left[-1\pm\left(1+\dfrac{2k(x+\gamma y)}{(1+\gamma^2)^2}\right)^{1/2}\right]$	$\dot{x}+F_3+\ln\lvert\dot{x}+\gamma\dot{y}+G_3\rvert$	0	G_3-ky/γ
V	$-ky$	$-k(1+\gamma^2)y$	$\dot{x}+F_3+\ln\lvert\dot{x}+\gamma\dot{y}+G_3\rvert$	k	$k(y-\gamma x)-\frac{1}{2}k^2(1+\gamma^2)y^2$
VI	$-y/\gamma-\exp(-y/\gamma)$	$-y/\gamma+x-(1+\gamma^2)\exp(-y/\gamma)$	$\dot{x}+F_3+\ln\lvert\dot{x}+\gamma\dot{y}+G_3\rvert$	$[1-\exp(-y/\gamma)]/\gamma$	$-\left(\dfrac{y}{\gamma}-x\right)\exp(-y/\gamma)-\frac{1}{2}(1+\gamma^2)\exp(-2y/\gamma)$
VII	$-\ln\lvert y\rvert$	$-\left(1+\dfrac{x^2}{y^2}\right)y+\left(k^2+c^2\dfrac{x^2}{y^2}\right)^{1/2}$	$\dot{x}+F_3+\ln\lvert\dot{x}y-x\dot{y}+G_3\rvert$	$1/y$	$-\dfrac{x^2+c^2}{2y^2}+\dfrac{(k^2y^2+c^2x^2)^{1/2}}{y^2}$

4. Comments and conclusions

In this paper seven examples of integrable velocity-dependent potentials which possess, in addition to the energy integral

$$E = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + U(x, y)$$

a second invariant with a logarithmic part in the velocity terms, are found. These examples are given in table 1.

Example I is related to the case (4.2*b*) of Hietarinta (1986, p 80) while, for $l = 0$, it corresponds to the system given by Hietarinta (1984).

Example III is also related to case (4.2*c*) given in Hietarinta (1986, p 81).

Example IV is of particular interest since it is the sole example of a velocity-independent potential possessing a logarithmic invariant. In this case, integral I does not have a good time reflection parity in the sense of Hietarinta (1986, p 13), while the Hamiltonian does. Thus the invariant I can be decomposed in the following way:

$$I_+ = \tfrac{1}{2}[I(t) + I(-t)] = \tfrac{1}{2}\ln|G_3^2 - (\dot{x} + \gamma\dot{y})^2|$$

$$I_- = \tfrac{1}{2}[I(t) - I(-t)] = \dot{x} + \tfrac{1}{2}\ln|[(G_3 + \dot{x} + \gamma\dot{y})/(G_3 - \dot{x} - \gamma\dot{y})|$$

where $I(-t)$ is I after time reflection. Both I_+ and I_- are independent integrals of motion with good time reflection parity, so the systems of example IV are superintegrable. A simpler realisation of I_+ is obviously

$$I_+ = G_3^2 - (\dot{x} + \gamma\dot{y})^2.$$

In figure 1, some of the orbits of this system are given for the special values $k = 2$, $\gamma = 1$ where the plus sign in G_3 has been selected. All orbits correspond to the initial conditions $x_0 = y_0 = 0$, $E = 4$ while $\dot{x}_0$ varies. The broken line $x + y = 0$ corresponds to the solution

$$(x - y) = -t^2 + (\dot{x}_0 - \dot{y}_0)t.$$

All the orbits above this solution tend to infinity, while all orbits below sink into the singularity $x + y = -1$.

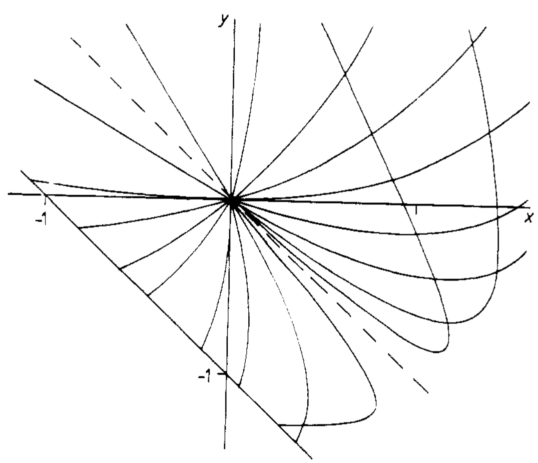

Figure 1. Orbits for the system of example IV for $k = 2$, $\gamma = 1$. All orbits correspond to initial conditions $x_0 = y_0 = 0$, $E = 4$.

There is no doubt that more complicated particular solutions of the equations given in this paper can also be obtained in each case, but the calculations may become tedious. It seems, however, that systems possessing an invariant, which includes a logarithmic term in the velocities, are not the exception among integrable velocity-dependent potentials.

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